

SIMILARITY OF QUOTIENT HILBERT MODULES IN THE COWEN-DOUGLAS CLASS

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ABSTRACT. In this paper, we consider the similarity problem for Hilbert modules in the Cowen-Douglas class associated with the complex geometric object, the hermitian anti-holomorphic vector bundles. Given a "simple" rank one Cowen-Douglas Hilbert module \mathcal{M} , we find necessary and sufficient conditions for the Cowen-Douglas Hilbert modules satisfying some positivity conditions to be similar to $\mathcal{M} \otimes \mathbb{C}^m$. We also show that under certain uniform bound condition on the anti-holomorphic frame, a class of Cowen-Douglas Hilbert modules are quasi-affinity to submodules of $\mathcal{M} \otimes \mathbb{C}^m$.

1. INTRODUCTION

One of the most challenging problems in operator theory is to determine when two given bounded linear operators are similar. More precisely, let T and R be two bounded linear operators on Hilbert spaces \mathcal{H} and \mathcal{K} , respectively. When does there exists an invertible bounded linear map $X : \mathcal{H} \rightarrow \mathcal{K}$ such that $XT = RX$?

There are many fascinating subtleties connected with the similarity problem (see [19], [18], [15], [12], [14]). However, the problem becomes more tractable if one impose additional assumptions on the operators. In particular, there are several characterizations of operators similar to unitaries or isometries or even contractions.

In [20], Uchiyama proposed a characterization for contractions in the Cowen-Douglas class which are similar to the adjoint of the multiplication operators on the Hardy space with finite multiplicity. One of the main tools used in the work by Uchiyama is the tensor product bundle corresponding to a given hermitian holomorphic vector bundle. Later, Kwon and Treil [13] found some new characterizations which involves the curvature, in the sense of Cowen-Douglas, and the Carleson measure [5] of the underlying operators. More recently, the work by Douglas, Kwon and Treil [8] improves the earlier results on contractions to the class of n -hypercontractions [1].

In the present study, we set up the similarity problem in a more general framework and also provide some characterizations of operators in the Cowen-Douglas class which are similar to the adjoint of the multiplication operators on the Hilbert spaces of holomorphic functions. More precisely, we prove that the earlier characterizations of operators similar to the adjoint of multiplication operators are valid beyond the class of contractions and n -hypercontractions. In particular, our results includes the similarity problem for the weighted Bergman spaces

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with not necessarily integer weights. Our framework is based on the $\frac{1}{K}$ -calculus, in the sense of Arazy and Englis [4], and the techniques involved in the proofs are essentially along the lines of [20], [17], [13] and [8].

The results of this paper, while similar to results obtained in [20], [13] and [8], are significantly more general.

We now summarize the content of this paper. We begin our presentation in Section 2 with an overview of terminology and notation. In Section 3, we obtain results concerning the tensor product bundles and quotient modules. In Section 4, we relate the curvatures to the derivatives of (anti) holomorphic maps. In Section 5, we present results concerning quasi-similarity and similarity. We conclude in Section 6 with a discussion of some possible directions in the study of similarity.

2. PRELIMINARIES

In this section we introduce the basic concepts and known results related to the Cowen-Douglas class and $\frac{1}{K}$ -calculus. Our presentation of $\frac{1}{K}$ -calculus is restricted to one variable. For more details, we refer the readers to the work by Arazy and Englis [4].

DEFINITION 2.1. *Let T be a linear operator on a Hilbert space \mathcal{H} . Then \mathcal{H} is said to be a Hilbert module over $\mathbb{C}[z]$ with the module action $\mathbb{C}[z] \times \mathcal{H}$ to \mathcal{H} given by*

$$p \cdot f \mapsto p(T)f,$$

for $p \in \mathbb{C}[z]$.

Note that the Hilbert module is uniquely determined by the underlying operator T via the module multiplication operator

$$M_z f := z \cdot f = Tf,$$

for all $f \in \mathcal{H}$ and vice versa.

A Hilbert module \mathcal{H} is said to be *contractive* if

$$I_{\mathcal{H}} - M_z M_z^* \geq 0.$$

We denote the space of all bounded linear operators from a Hilbert space \mathcal{H} to another Hilbert space \mathcal{K} by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, and by $\mathcal{B}(\mathcal{H})$ if $\mathcal{K} = \mathcal{H}$.

Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert modules over $\mathbb{C}[z]$. Then $M \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be a module map if $M(p \cdot f) = p \cdot (Mf)$ for all $p \in \mathbb{C}[z]$ and $f \in \mathcal{H}_1$.

Now we recall the definition of the Cowen-Douglas class [6].

DEFINITION 2.2. *Let m be a positive integer. The Cowen-Douglas class on \mathbb{D} of rank m , denoted by $B_m(\mathbb{D})$ is the set of all Hilbert modules \mathcal{H} over $\mathbb{C}[z]$ such that*

- (i) $\sigma(M_z^*) \subseteq \mathbb{D}$,
- (ii) $\text{ran}(M_z - wI_{\mathcal{H}})^* = \mathcal{H}$ for all $w \in \mathbb{D}$,
- (iii) $\dim \ker(M_z - wI_{\mathcal{H}})^* = m$ for all $w \in \mathbb{D}$, and
- (iv) $\overline{\text{span}}\{\ker(M_z - wI_{\mathcal{H}})^* : w \in \mathbb{D}\} = \mathcal{H}$.

A Hilbert module \mathcal{H} is said to be a *Cowen-Douglas Hilbert module* if $\mathcal{H} \in B_m(\mathbb{D})$ for some positive integer m .

Conditions (ii) and (iii) in the above definition implies that for $\mathcal{H} \in B_m(\mathbb{D})$, $(M_z - wI_{\mathcal{H}})^*$ is Fredholm and $\text{ind}(M_z - wI_{\mathcal{H}})^* = m$ for all $w \in \mathbb{D}$.

A Cowen-Douglas Hilbert module \mathcal{H} in $B_m(\mathbb{D})$ defines a hermitian anti-holomorphic vector bundle $E_{\mathcal{H}}$ over \mathbb{D} where

$$E_{\mathcal{H}} = \{(w, h) \in \mathbb{D} \times \mathcal{H} : M_z^* h = \bar{w}h\},$$

with the projection map $\pi_{\mathcal{H}} : E_{\mathcal{H}} \rightarrow \mathbb{D}$ defined by $\pi_{\mathcal{H}}(w, h) = w$ for all $w \in \mathbb{D}$ and $h \in \mathcal{H}$. More precisely, $E_{\mathcal{H}}$ is the anti-holomorphic vector bundle implemented by the anti-holomorphic map $w \mapsto E_{\mathcal{H}}(w) := \ker (M_z - wI_{\mathcal{H}})^*$, the pull-back bundle of the Grassmannian $GF(m, \mathcal{H})$ (see [6]) and hence locally at each point $w \in \mathbb{D}$, there exists anti-holomorphic \mathcal{H} -valued functions $\{\gamma_{i,w} : 1 \leq i \leq m\}$ such that

$$\text{span}\{\gamma_{i,w} : 1 \leq i \leq m\} = \ker(M_z - w)^*.$$

Also it follows from a theorem of Grauert [11] that $\gamma_{i,w}$ can be defined on all of \mathbb{D} .

The rigidity theorem (Theorem 2.2 in [6]) states that a pair of Hilbert modules \mathcal{H} and $\tilde{\mathcal{H}}$ in $B_m(\mathbb{D})$ are unitarily equivalent if and only if the corresponding hermitian anti-holomorphic vector bundles $E_{\mathcal{H}}$ and $E_{\tilde{\mathcal{H}}}$ are equivalent.

Let \mathcal{E} be a Hilbert space. A Hilbert module $\mathcal{H} \subseteq \mathcal{O}(\mathbb{D}, \mathcal{E})$, where $\mathcal{O}(\mathbb{D}, \mathcal{E})$ is the space of \mathcal{E} -valued holomorphic functions on \mathbb{D} is said to be a *reproducing kernel Hilbert module* if (i) the evaluation map $ev_w : \mathcal{H} \rightarrow \mathcal{E}$ defined by $ev_w(f) = f(w)$ is bounded for all $w \in \mathbb{D}$, and (ii) the module multiplication operator M_z is given by the multiplication operator by the coordinate function z .

The kernel function of a reproducing kernel Hilbert module is given by

$$k(z, w) = ev_z \circ ev_w^*,$$

for all $z, w \in \mathbb{D}$. Note that the kernel function $k(z, w)$ is holomorphic in z and anti-holomorphic in w .

We point out that a Hilbert module \mathcal{H} in $B_m(\mathbb{D})$ is unitarily equivalent to a reproducing kernel Hilbert module \mathcal{H}_k for some kernel function $k : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{B}(\mathbb{C}^m)$ (see [2], [7]).

Let $\mathcal{H} \in B_m(\mathbb{D})$ with an anti-holomorphic frame $\{\gamma_{i,w} : 1 \leq i \leq m\}$ of $E_{\mathcal{H}}$. The curvature matrix of $E_{\mathcal{H}}$ is given by

$$\mathcal{K}_{\mathcal{H}}(w) = -\bar{\partial}[G^{-1}\partial G],$$

for all $w \in \mathbb{D}$, where

$$G(w) = (\langle \gamma_{i,w}, \gamma_{j,w} \rangle_{E_{\mathcal{H}}(w)})_{i,j=1}^m,$$

is the Gram matrix corresponding to the anti-holomorphic frame $\{\gamma_{i,w} : 1 \leq i \leq m\}$. In particular, if $E_{\mathcal{H}}$ is a line bundle then

$$\mathcal{K}_{\mathcal{H}}(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log \|\gamma_w\|^2.$$

Our results are based on the following class of Cowen-Douglas Hilbert modules of rank one.

DEFINITION 2.3. A Hilbert module $\mathcal{M} \in B_1(\mathbb{D})$ is said to be a Cowen-Douglas atom if

- (i) the set of polynomials $\mathbb{C}[z]$ is dense in \mathcal{M} ,
- (ii) there exists a sequence of polynomials $\{p_l(z, \bar{w}) : z, \bar{w} \in \mathbb{D}\}$ such that

$$p_l(z, \bar{w}) \rightarrow \frac{1}{k_{\mathcal{M}}(z, w)},$$

as $l \rightarrow \infty$ and for all $z, \bar{w} \in \mathbb{D}$ and

- (iii) $\sup_l \|p_l(M_z, M_z^*)\| < \infty$, and
- (iv) $\{M_z\}' = H^\infty(\mathbb{D})$.

Appealing to Theorem 1.6 of [4], a Cowen-Douglas atom \mathcal{M} admits a $\frac{1}{K}$ -calculus. Here we do not intend to define the $\frac{1}{K}$ -calculus but spell out the required properties of such concept in the present set up. We again refer the reader to [4] for details.

Note that by condition (i) in the above definition and the Gram-Schmidt orthogonalization process, for a Cowen-Douglas atom \mathcal{M} there exists a sequence of orthonormal basis of polynomials $\{q_l(z) : l \geq 0\}$ and

$$(2.1) \quad k_{\mathcal{M}}(z, w) = \sum_{l \geq 0} q_l(z) \overline{q_l(w)}.$$

We henceforth assume \mathcal{M} to be a fixed Cowen-Douglas atom with the sequence of polynomials $\{p_l(z, \bar{w})\}$ as in (ii) of Definition 2.3 and the orthonormal basis as above with the kernel function identity (2.1).

Natural examples of Cowen-Douglas atoms include the Hardy space and the weighted Bergman spaces.

We turn now to define an analogue of the contractive Hilbert modules.

DEFINITION 2.4. A Hilbert module \mathcal{H} over $\mathbb{C}[z]$ is said to be \mathcal{M} -contractive if

- (i) $\sup_l \|p_l(M_z, M_z^*)\| < \infty$ and
- (ii) $C_{\mathcal{H}} := \text{WOT} - \lim_{l \rightarrow \infty} p_l(M_z, M_z^*)$ is a positive operator.

Let A be a positive operator on a Hilbert space \mathcal{H} . Define $Q_{l,A} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ for all $l \geq 1$ by

$$Q_{l,A}(T) = I - \sum_{0 \leq j < l} q_j(T) A q_j(T)^*,$$

for all $T \in \mathcal{B}(\mathcal{H})$.

An \mathcal{M} -contractive Hilbert module \mathcal{H} is said to be *pure* if

$$\text{SOT} - \lim_{l \rightarrow \infty} Q_{l,C_{\mathcal{H}}}(M_z) = 0.$$

3. QUOTIENT MODULES AND TENSOR PRODUCT BUNDLES

The aim of the present section is to prove that the hermitian anti-holomorphic vector bundle of a pure \mathcal{M} -contractive Hilbert module in $B_m(\mathbb{D})$ can be represented as the tensor product bundle of a hermitian anti-holomorphic line bundle and a rank m hermitian anti-holomorphic vector bundle.

We start by recalling a version of the model theorem due to Arazy and Englis (Corollary 3.2 in [4]).

THEOREM 3.1. (Arazy-Englis) *Let \mathcal{H} be a Hilbert module over $\mathbb{C}[z]$. Then \mathcal{H} is a pure \mathcal{M} -contractive Hilbert module if and only if \mathcal{H} is unitarily equivalent to a quotient module of $\mathcal{M} \otimes \mathcal{E}$ for some Hilbert space \mathcal{E} .*

The following proposition shows that the \mathcal{M} -contractive Hilbert modules in $B_m(\mathbb{D})$ are pure.

PROPOSITION 3.2. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be an \mathcal{M} -contractive Hilbert module. Then \mathcal{H} is pure.*

Proof. Let $\{\gamma_{i,w} : 1 \leq i \leq m\}$ be an anti-holomorphic frame of $E_{\mathcal{H}}$ with

$$M_z^* \gamma_{i,w} = \bar{w} \gamma_{i,w},$$

for all $w \in \mathbb{D}$ and $1 \leq i \leq m$. Then for all $z, w \in \mathbb{D}$ and $1 \leq i, j \leq m$ we have

$$\langle C_{\mathcal{H}} \gamma_{i,w}, \gamma_{j,z} \rangle = \lim_{l \rightarrow \infty} \langle p_l(M_z, M_z^*) \gamma_{i,w}, \gamma_{j,z} \rangle = \left(\lim_{l \rightarrow \infty} p_l(z, \bar{w}) \right) \langle \gamma_{i,w}, \gamma_{j,z} \rangle = \frac{1}{k_{\mathcal{M}}(z, w)} \langle \gamma_{i,w}, \gamma_{j,z} \rangle,$$

and hence

$$\begin{aligned} \langle Q_{l, C_{\mathcal{H}}}(M_z) \gamma_{i,w}, \gamma_{j,z} \rangle &= \langle \gamma_{i,w}, \gamma_{j,z} \rangle - \left\langle \sum_{t=0}^{l-1} q_t(M_z) C_{\mathcal{H}} q_t(M_z)^* \gamma_{i,w}, \gamma_{j,z} \right\rangle \\ &= \langle \gamma_{i,w}, \gamma_{j,z} \rangle - \left\langle \sum_{t=0}^{l-1} C_{\mathcal{H}} \overline{q_t(w)} \gamma_{i,w}, \overline{q_t(z)} \gamma_{j,z} \right\rangle \\ &= \langle \gamma_{i,w}, \gamma_{j,z} \rangle - \left(\sum_{t=0}^{l-1} q_t(z) \overline{q_t(w)} \right) \langle C_{\mathcal{H}} \gamma_{i,w}, \gamma_{j,z} \rangle \\ &= \langle \gamma_{i,w}, \gamma_{j,z} \rangle - \left(\sum_{t=0}^{l-1} q_t(z) \overline{q_t(w)} \right) \frac{1}{k_{\mathcal{M}}(z, w)} \langle \gamma_{i,w}, \gamma_{j,z} \rangle \\ &= \left(1 - \left(\sum_{t=0}^{l-1} q_t(z) \overline{q_t(w)} \right) \frac{1}{k_{\mathcal{M}}(z, w)} \right) \langle \gamma_{i,w}, \gamma_{j,z} \rangle \\ &\rightarrow 0 \text{ as } l \rightarrow \infty. \end{aligned}$$

From this we deduce that $Q_{l, C_{\mathcal{H}}}(M_z) \rightarrow 0$ in SOT. This concludes the proof. \blacksquare

Let $\mathcal{H} \in B_m(\mathbb{D})$ be an \mathcal{M} -contractive module. As an application of the previous proposition and Theorem 3.1, \mathcal{H} can be realized as $\mathcal{H} \cong \mathcal{Q} := (\mathcal{M} \otimes \mathcal{E}) / \mathcal{S}$ for some submodule \mathcal{S} of $\mathcal{M} \otimes \mathcal{E}$ and a Hilbert space \mathcal{E} . That is,

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{M} \otimes \mathcal{E} \rightarrow \mathcal{H} \rightarrow 0.$$

Therefore, an \mathcal{M} -contractive Hilbert module $\mathcal{H} \in B_m(\mathbb{D})$ can be realized as a quotient module \mathcal{Q} of $\mathcal{M} \otimes \mathcal{E}$ for some coefficient space \mathcal{E} . In this representation, the module map M_z on \mathcal{H} is identified with the compressed multiplication operator $P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}$. Moreover,

$$P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})^*|_{\mathcal{Q}} = (M_z \otimes I_{\mathcal{E}})^*|_{\mathcal{Q}}.$$

In the rest of this paper we will assume the quotient module representations of the class of pure \mathcal{M} -contractive Hilbert modules in $B_m(\mathbb{D})$.

Also, we will identify a Cowen-Douglas atom \mathcal{M} with the reproducing kernel Hilbert module with section $w \mapsto k_{\mathcal{M}}(\cdot, w)$ for all $w \in \mathbb{D}$.

The following lemma will be very useful in the sequel.

LEMMA 3.3. *Let \mathcal{M} be a Cowen-Douglas atom and \mathcal{E} be a Hilbert space. Also let $\mathcal{Q} = (\mathcal{M} \otimes \mathcal{E})/\mathcal{S}$ be an \mathcal{M} -contractive Hilbert module for some submodule \mathcal{S} of $\mathcal{M} \otimes \mathcal{E}$. Then,*

- (i) $C_{\mathcal{M} \otimes \mathcal{E}} = I_{\mathcal{M}} \otimes P_{\mathcal{E}}$, and
- (ii) $C_{\mathcal{Q}} = P_{\mathcal{Q}} C_{\mathcal{M} \otimes \mathcal{E}} P_{\mathcal{Q}} = P_{\mathcal{Q}} (I_{\mathcal{M}} \otimes P_{\mathcal{E}}) P_{\mathcal{Q}}$.

Proof. Let $z, w \in \mathbb{D}$ and $x, y \in \mathcal{E}$. Then for all $l \geq 1$ we have

$$\begin{aligned} \langle p_l(M_z \otimes I_{\mathcal{E}}, M_z^* \otimes I_{\mathcal{E}})(k_{\mathcal{M}}(\cdot, w) \otimes x), k_{\mathcal{M}}(\cdot, z) \otimes y \rangle &= p_l(z, \bar{w}) \langle k_{\mathcal{M}}(\cdot, w) \otimes x, k_{\mathcal{M}}(\cdot, z) \otimes y \rangle \\ &= p_l(z, \bar{w}) k_{\mathcal{M}}(z, w) \langle x, y \rangle. \end{aligned}$$

Consequently, by letting $l \rightarrow \infty$, we obtain

$$\begin{aligned} \langle C_{\mathcal{M} \otimes \mathcal{E}}(k_{\mathcal{M}}(\cdot, w) \otimes x), k_{\mathcal{M}}(\cdot, z) \otimes y \rangle &= \frac{1}{k_{\mathcal{M}}(z, w)} k_{\mathcal{M}}(z, w) \langle x, y \rangle = \langle x, y \rangle \\ &= \langle (I_{\mathcal{M}} \otimes P_{\mathcal{E}})(k_{\mathcal{M}}(\cdot, w) \otimes x), k_{\mathcal{M}}(\cdot, z) \otimes y \rangle. \end{aligned}$$

This completes the proof of part (i).

To prove (ii) we compute

$$p_l(P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}, P_{\mathcal{Q}}(M_z^* \otimes I_{\mathcal{E}})|_{\mathcal{Q}}) = P_{\mathcal{Q}}(p_l(M_z \otimes I_{\mathcal{E}}, M_z^* \otimes I_{\mathcal{E}}))|_{\mathcal{Q}}.$$

Letting $l \rightarrow \infty$ in WOT, we deduce from part (i) that

$$C_{\mathcal{Q}} = P_{\mathcal{Q}}(I \otimes P_{\mathcal{E}})|_{\mathcal{Q}}.$$

This completes the proof of the lemma. ■

Now we are in a position to prove the main result of this section.

THEOREM 3.4. *Let $\mathcal{Q} \in B_m(\mathbb{D})$ be a pure \mathcal{M} -contractive Hilbert module given by*

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{M} \otimes \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0,$$

where \mathcal{S} is a submodule of $\mathcal{M} \otimes \mathcal{E}$ and \mathcal{E} is a coefficient Hilbert space. Then there exists a rank n hermitian anti-holomorphic vector bundle V over \mathbb{D} such that

$$E_{\mathcal{Q}} \cong E_{\mathcal{M}} \otimes V.$$

Moreover,

$$\mathcal{K}_{\mathcal{Q}} = \mathcal{K}_{\mathcal{M}} + \mathcal{K}_V.$$

Proof. Let $\{\gamma_{i,w} : 1 \leq i \leq m\}$ be an anti-holomorphic frame of $E_{\mathcal{Q}}$ such that

$$M_z^* \gamma_{i,w} = \bar{w} \gamma_{i,w},$$

for all $1 \leq i \leq m$ and $w \in \mathbb{D}$. Then for all $l \geq 1$ we have

$$p_l(M_z, M_z^*) \gamma_{i,w} = p_l(z, \bar{w}) \gamma_{i,w}.$$

Letting $l \rightarrow \infty$ in WOT, and applying Lemma 3.3 we have

$$\frac{1}{k_{\mathcal{M}}(\cdot, w)}\gamma_{i,w} = C_{\mathcal{Q}}\gamma_{i,w} = P_{\mathcal{Q}}(I \otimes P_{\mathcal{E}})\gamma_{i,w}.$$

Since

$$P_{\mathcal{Q}}(I \otimes P_{\mathcal{E}})\gamma_{i,w} = \gamma_{i,w}(0),$$

we have

$$\frac{1}{k_{\mathcal{M}}(\cdot, w)}\gamma_{i,w} = \gamma_{i,w}(0).$$

Therefore

$$(3.1) \quad \gamma_{i,w} = k_{\mathcal{M}}(\cdot, w) \otimes \gamma_{i,w}(0) = k_{\mathcal{M}}(\cdot, w) \otimes v_{i,w},$$

where $v_{i,w} := \gamma_{i,w}(0)$ for all $1 \leq i \leq m$ and $w \in \mathbb{D}$. Let V be the anti-holomorphic curve over \mathbb{D} with $V(w) = \text{span} \{v_{i,w} : 1 \leq i \leq m\}$. Then we conclude that $E_{\mathcal{Q}} \cong E_{\mathcal{M}} \otimes V$.

Finally, let G_V be the Gram matrix corresponding to the frame $\{v_{i,w}\}$ of E_V . Then

$$\begin{aligned} \mathcal{K}_{E_{\mathcal{Q}}}(w) &= -\bar{\partial}[G_{E_{\mathcal{M}}}^{-1} \partial G_{E_{\mathcal{M}}}] = -\bar{\partial}\left[\frac{1}{\|k_{\mathcal{M}}(\cdot, w)\|^2} G_V^{-1} \partial\{\|k_{\mathcal{M}}(\cdot, w)\|^2 G_V\}\right] \\ &= -\bar{\partial}\left[\frac{1}{\|k_{\mathcal{M}}(\cdot, w)\|^2} G_V^{-1} \{\partial(\|k_{\mathcal{M}}(\cdot, w)\|^2) G_V + \|k_{\mathcal{M}}(\cdot, w)\|^2 \partial G_V\}\right] \\ &= -\bar{\partial}\left[\frac{1}{\|k_{\mathcal{M}}(\cdot, w)\|^2} \partial(\|k_{\mathcal{M}}(\cdot, w)\|^2) + G_V^{-1} \partial(G_V)\right] \\ &= \mathcal{K}_{E_{\mathcal{M}}}(w) + \mathcal{K}_V(w). \end{aligned}$$

This concludes the proof of the theorem. ■

4. DERIVATIVES OF HOLOMORPHIC MAPS AND CURVATURES

In this section we obtain the curvature of a hermitian anti-holomorphic line bundle as the derivative of the orthogonal projection of the corresponding projection map. We begin with some identities of the derivatives of projections.

For $\mathcal{H} \in B_m(\mathbb{D})$, we denote the orthogonal projection

$$\Pi(w) = P_{\ker (M_z - w)^*},$$

for all $w \in \mathbb{D}$. Then there exists (locally) an anti-holomorphic $\mathcal{B}(\mathbb{C}^m, \mathcal{H})$ -valued function Γ such that $\text{ran } \Gamma(w) = \ker (M_z - w)^*$ (see Theorem 2.2 in [7]). Since Γ is left invertible, we have

$$\Pi = \Gamma(\Gamma^* \Gamma)^{-1} \Gamma^*.$$

It is well known that (cf. [17])

$$(4.1) \quad \Pi \bar{\partial} \Pi = 0,$$

and

$$(4.2) \quad (\bar{\partial} \Pi) \Pi = \bar{\partial} \Pi,$$

and

$$(4.3) \quad (I - \Pi)(\bar{\partial}\Pi)\Pi = \bar{\partial}\Pi.$$

In the sequel, we denote by $\|\cdot\|_2$, the Hilbert-Schmidt norm of operators.

The following proposition is a generalization of earlier results on the Hardy space and the weighted Bergman spaces with integer weights (see Lemma 1.7 in [13] and Lemma 3.3 in [8]). The result may be known in the study of complex geometry (see Remark 0.3 in [13]), but as we have been unable to locate any explicit proof, we provide full details for the sake of completeness.

PROPOSITION 4.1. *Let $\mathcal{H} \in B_1(\mathbb{D})$ and $\Pi(w) = P_{\ker(M_z - w)^*}$ and $\mathcal{K}_{\mathcal{H}}$ be the curvature of \mathcal{H} . Then*

$$\|\bar{\partial}\Pi(w)\|_2^2 = |\mathcal{K}_{\mathcal{H}}(w)|,$$

for all $w \in \mathbb{D}$.

Proof. Since $\Pi(w)$ is a rank one projection onto $k_{\mathcal{M}}(\cdot, w)\mathbb{C}$, we have

$$\Pi(w)f = \frac{f(w)}{\|k(\cdot, w)\|^2} k(\cdot, w),$$

for all $f \in \mathcal{H}$. We have then for all $w \in \mathbb{D}$,

$$\Pi(w) = ev_w \frac{k(\cdot, w)}{\|k(\cdot, w)\|^2},$$

where ev_w is the evaluation functional at w defined by $ev_w(f) = f(w)$ for all $f \in \mathcal{H}$. Since $w \rightarrow ev_w$ is a holomorphic map it follows that

$$\bar{\partial}\Pi(w) = ev_w \bar{\partial}\left(\frac{k(\cdot, w)}{\|k(\cdot, w)\|^2}\right).$$

Now

$$\begin{aligned} \bar{\partial}\left(\frac{k(\cdot, w)}{\|k(\cdot, w)\|^2}\right) &= \frac{k'(\cdot, w)}{\|k(\cdot, w)\|^2} - \frac{k(\cdot, w)}{\|k(\cdot, w)\|^4} \bar{\partial}(\|k(\cdot, w)\|^2) \\ &= \frac{k'(\cdot, w)}{\|k(\cdot, w)\|^2} - \frac{k(\cdot, w)}{\|k(\cdot, w)\|^4} \langle k'(\cdot, w), k(\cdot, w) \rangle \\ &= \frac{1}{\|k(\cdot, w)\|^4} (\|k(\cdot, w)\|^2 k'(\cdot, w) - \langle k'(\cdot, w), k(\cdot, w) \rangle k(\cdot, w)). \end{aligned}$$

It follows that (or see page 195 in [6])

$$\begin{aligned} &\| \|k(\cdot, w)\|^2 k'(\cdot, w) - \langle k'(\cdot, w), k(\cdot, w) \rangle k(\cdot, w) \| \\ &= \|k(\cdot, w)\| (\|k(\cdot, w)\|^2 \|k'(\cdot, w)\|^2 - |\langle k'(\cdot, w), k(\cdot, w) \rangle|^2)^{\frac{1}{2}}. \end{aligned}$$

Thus the last equality and $\|ev_w\| = \|k(\cdot, w)\|$ imply that

$$\begin{aligned}\|\bar{\partial}\Pi(w)\|_2^2 &= \|ev_w\|^2 \|\bar{\partial}(\frac{k(\cdot, w)}{\|k(\cdot, w)\|^2})\|^2 \\ &= (\frac{1}{\|k(\cdot, w)\|^2} \{\|k(\cdot, w)\|^2 \|k'(\cdot, w)\|^2 - |\langle k'(\cdot, w), k(\cdot, w) \rangle|^2\}^{\frac{1}{2}})^2 \\ &= |\mathcal{K}_{\mathcal{H}}(w)|.\end{aligned}$$

This completes the proof of the proposition. \blacksquare

Let \mathcal{H} be a Hilbert space and let $GF(m, \mathcal{H})$ be the Grassmannian manifold of all m -dimensional subspaces of \mathcal{H} . Given an anti-holomorphic curve $\varphi : \mathbb{D} \rightarrow GF(m, \mathcal{H})$ one can define the hermitian anti-holomorphic vector bundle E with the fibre $E(w) = \varphi(w)$ for all $w \in \mathbb{D}$. Let $\Pi_E : \mathbb{D} \rightarrow \mathcal{B}(\mathcal{H})$ be the anti-holomorphic map defined by

$$\Pi_E(w) = P_{E(w)},$$

where $P_{E(w)}$ is the orthogonal projection onto $E(w)$.

COROLLARY 4.2. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be an \mathcal{M} -contractive Hilbert module. Then there exists a hermitian anti-holomorphic vector bundle V of rank m over \mathbb{D} such that $E_{\mathcal{H}} \cong E_{\mathcal{M}} \otimes V$. Moreover,*

$$\Pi_{\mathcal{H}} = \Pi_{\mathcal{M}} \otimes \Pi_V,$$

and for all $w \in \mathbb{D}$,

$$\|\bar{\partial}\Pi_{\mathcal{H}}(w)\|_2^2 = m\|\bar{\partial}\Pi_{\mathcal{M}}(w)\|_2^2 + \|\bar{\partial}\Pi_V(w)\|_2^2 = m|\mathcal{K}_{\mathcal{M}}(w)| + \|\bar{\partial}\Pi_V(w)\|_2^2.$$

Proof. First two conclusions follows from Theorem 3.4. For the remaining parts, we follow the same line of arguments as in [13] or [8]. Since

$$\bar{\partial}\Pi_{\mathcal{Q}}(w) = \bar{\partial}(\Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)) = \bar{\partial}\Pi_{\mathcal{M}}(w) \otimes \Pi_V(w) + \Pi_{\mathcal{M}}(w) \otimes \bar{\partial}\Pi_V(w),$$

we have that

$$\begin{aligned}\|\bar{\partial}\Pi_{\mathcal{Q}}(w)\|_2^2 &= \text{tr}([\bar{\partial}\Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)][\bar{\partial}\Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)]^*) + 2\text{Real}\{\text{tr}([\bar{\partial}\Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)]^* \\ &\quad [\Pi_{\mathcal{M}}(w) \otimes \bar{\partial}\Pi_V(w)])\} + \text{tr}([\Pi_{\mathcal{M}}(w) \otimes \bar{\partial}\Pi_V(w)]^*[\Pi_{\mathcal{M}}(w) \otimes \bar{\partial}\Pi_V(w)]).\end{aligned}$$

By (4.1) we have $\bar{\partial}\Pi_{\mathcal{M}}(w)\Pi_{\mathcal{M}}(w) = 0$ and hence the middle term in the last expression vanishes. Therefore,

$$\begin{aligned}\|\bar{\partial}\Pi_{\mathcal{Q}}(w)\|_2^2 &= \|\bar{\partial}\Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)\|_2^2 + \|\Pi_{\mathcal{M}}(w) \otimes \bar{\partial}\Pi_V(w)\|_2^2 \\ &= \|\bar{\partial}\Pi_{\mathcal{M}}(w)\|_2^2 \|\Pi_V(w)\|_2^2 + \|\Pi_{\mathcal{M}}(w)\|_2^2 \|\bar{\partial}\Pi_V(w)\|_2^2 \\ &= m|\mathcal{K}_{\mathcal{M}}(w)| + \|\bar{\partial}\Pi_V(w)\|_2^2,\end{aligned}$$

where the last equality follows from Proposition 4.1. This completes the proof. \blacksquare

5. QUASI-AFFINITY AND SIMILARITY

In this section we discuss the issues of quasi-affinity and similarity of Hilbert modules in the Cowen-Douglas class $B_m(\mathbb{D})$.

Let \mathcal{H} and \mathcal{K} be two Hilbert modules. Then we say that \mathcal{H} is *quasi-affine* to \mathcal{K} , denoted by $\mathcal{H} \prec \mathcal{K}$, if there exists a module map $X : \mathcal{H} \rightarrow \mathcal{K}$ such that X is one-to-one and has dense range.

THEOREM 5.1. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be an \mathcal{M} -contractive Hilbert module and $\{\gamma_{i,w}\}_{i=1}^m$ be an anti-holomorphic frame of $E_{\mathcal{H}}$ such that*

$$\sup_{w \in \mathbb{D}} \left(\frac{\|\gamma_{i,w}\|}{\|k_{\mathcal{M}}(\cdot, w)\|} \right) < \infty,$$

for all $i = 1, \dots, m$. Then

- (i) *there exists a one-to-one module map $X : \mathcal{H} \rightarrow \mathcal{M} \otimes \mathbb{C}^m$, and*
- (ii) *$\mathcal{H} \prec \mathcal{S}$ for some submodule $\mathcal{S} \subseteq \mathcal{M} \otimes \mathbb{C}^m$.*

Proof. Identifying \mathcal{H} with $\mathcal{Q} = (\mathcal{M} \otimes \mathcal{E})/\mathcal{S}$ for some submodule \mathcal{S} of $\mathcal{M} \otimes \mathcal{E}$, we let

$$\gamma_{i,w} = k_{\mathcal{M}}(\cdot, w) \otimes v_{i,w},$$

with

$$\delta := \sup_{w \in \mathbb{D}} \left(\frac{\|\gamma_{i,w}\|}{\|k_{\mathcal{M}}(\cdot, w)\|} \right) = \sup_{w \in \mathbb{D}} \|v_{i,w}\| < \infty,$$

for all $i = 1, \dots, m$. For each $z \in \mathbb{D}$, define $\Theta(z) \in \mathcal{B}(\mathcal{E}, \mathbb{C}^m)$ by

$$\Theta(z)\eta = (\langle \eta, v_{1,z} \rangle_{\mathcal{E}}, \dots, \langle \eta, v_{m,z} \rangle_{\mathcal{E}}).$$

Then for all $\eta \in \mathcal{E}$ we have

$$\|\Theta(z)\eta\|^2 = \sum_{i=1}^m |\langle \eta, v_{i,z} \rangle_{\mathcal{E}}|^2 \leq \|\eta\|^2 \sum_{i=1}^m \|v_{i,w}\|^2 \leq m\delta^2 \|\eta\|^2.$$

Consequently, $\Theta \in H_{\mathcal{B}(\mathcal{E}, \mathbb{C}^m)}^{\infty}(\mathbb{D})$. Furthermore, for $f \in \mathcal{S} = \mathcal{Q}^{\perp}$ and $w \in \mathbb{D}$ we have

$$\begin{aligned} (\Theta f)(w) &= \Theta(w)f(w) = (\langle f(w), v_{1,z} \rangle_{\mathcal{E}}, \dots, \langle f(w), v_{m,z} \rangle_{\mathcal{E}}) \\ &= (\langle f, k_{\mathcal{M}}(\cdot, w) \otimes v_{1,z} \rangle_{\mathcal{M} \otimes \mathcal{E}}, \dots, \langle f, k_{\mathcal{M}}(\cdot, w) \otimes v_{m,z} \rangle_{\mathcal{M} \otimes \mathcal{E}}) \\ &= (\langle f, \gamma_{1,z} \rangle_{\mathcal{M} \otimes \mathcal{E}}, \dots, \langle f, \gamma_{m,z} \rangle_{\mathcal{M} \otimes \mathcal{E}}) \\ &= 0. \end{aligned}$$

Hence, $M_{\Theta}\mathcal{S} = \{0\}$. Next we define $X : \mathcal{Q} \rightarrow \mathcal{M} \otimes \mathbb{C}^m$ by

$$Xf = M_{\Theta}f,$$

for all $f \in \mathcal{Q}$. Then

$$\begin{aligned} XP_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}} &= M_{\Theta}P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}} = M_{\Theta}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}} = (M_z \otimes I_{\mathbb{C}^m})M_{\Theta}|_{\mathcal{Q}} \\ &= (M_z \otimes I_{\mathbb{C}^m})X, \end{aligned}$$

that is, X is a module map. To prove that X is one-to-one, or equivalently, that X^* has dense range, we compute

$$\langle \Theta(w)^* e_i, \eta \rangle_{\mathcal{E}} = \langle e_i, \Theta(w) \eta \rangle_{\mathbb{C}^m} = \langle e_i, \sum_{j=1}^m \langle \eta, v_{j,w} \rangle_{\mathcal{E}} e_j \rangle_{\mathbb{C}^m} = \langle v_{i,w}, \eta \rangle_{\mathcal{E}},$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{E}$. Therefore, $\Theta(w)^* e_i = v_{i,w}$ and hence

$$\begin{aligned} X^*(k_{\mathcal{M}}(\cdot, w) \otimes e_i) &= P_{\mathcal{Q}} M_{\Theta}^*(k_{\mathcal{M}}(\cdot, w) \otimes e_i) = P_{\mathcal{Q}}(k_{\mathcal{M}}(\cdot, w) \otimes \Theta(w)^* e_i) \\ &= (k_{\mathcal{M}}(\cdot, w) \otimes v_{i,w}) = P_{\mathcal{Q}} \gamma_{i,w} \\ &= \gamma_{i,w}. \end{aligned}$$

Hence, X is one-to-one. This proves part (i).

Part (ii) follows from part (i) and by considering \mathcal{S} as the range closure of X . \blacksquare

In the anti-holomorphic vector bundle language, the above result can be stated as follows : Suppose there exists an anti-holomorphic bundle map $\Phi : E_{\mathcal{M} \otimes \mathbb{C}^m} \rightarrow E_{\mathcal{H}}$ such that for some $\delta > 0$ we have

$$\|\Phi(w) \eta_w\|_{\mathcal{H}} \leq \delta \|\eta_w\|_{E_{\mathcal{M} \otimes \mathbb{C}^m}(w)},$$

for all $\eta_w \in E_{\mathcal{M} \otimes \mathbb{C}^m}(w)$ and $w \in \mathbb{D}$. Then \mathcal{H} is quasi-affine to a submodule of $\mathcal{M} \otimes \mathcal{E}$.

One might expect that the submodule \mathcal{S} in the above result is the entire free module $\mathcal{M} \otimes \mathbb{C}^m$. However, such results are closely related with the issue of the Beurling-Lax-Halmos type theorem for the Cowen-Douglas atoms. In particular, for $\mathcal{M} = H^2(\mathbb{D})$ the submodule \mathcal{S} is unitarily equivalent with the Hardy module with the same multiplicity as the rank of the map $\Theta(w)$ which is m . Consequently, for any $H^2(\mathbb{D})$ -contractive modules, the conclusion is stronger, that is, \mathcal{H} is quasi-affine to the Hardy module $H^2(\mathbb{D}) \otimes \mathbb{C}^m$ (see [20]). We point out that even the Bergman module is quite subtle [3] for this consideration.

The following result is a generalization of Theorem 3.8 in [20] which follows from Theorem 5.3. However, we supply a direct proof.

THEOREM 5.2. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be an \mathcal{M} -contractive Hilbert module. Then \mathcal{H} is similar to $\mathcal{M} \otimes \mathbb{C}^m$ if and only if there exists an anti-holomorphic pointwise invertible bundle map $\Phi : E_{\mathcal{M} \otimes \mathbb{C}^m} \rightarrow E_{\mathcal{H}}$ and $\delta > 0$ such that*

$$\frac{1}{\delta} \|\eta_w\|_{E_{\mathcal{M} \otimes \mathbb{C}^m}(w)} \leq \|\Phi(w) \eta_w\|_{\mathcal{H}} \leq \delta \|\eta_w\|_{E_{\mathcal{M} \otimes \mathbb{C}^m}(w)},$$

for all $\eta_w \in E_{\mathcal{M} \otimes \mathbb{C}^m}(w)$ and $w \in \mathbb{D}$.

Proof. Let $X : \mathcal{H} \rightarrow \mathcal{M} \otimes \mathbb{C}^m$ be an invertible module map. Then $\gamma_{i,w} := X^*(k_{\mathcal{M}}(\cdot, w) \otimes e_i)$ is the required anti-holomorphic frame of $E_{\mathcal{H}}$.

For the converse, we proceed as in the proof of Theorem 5.1. We first, consider an anti-holomorphic frame $\{\gamma_{i,w}\}_{i=1}^m = \{k_{\mathcal{M}}(\cdot, w) \otimes v_{i,w}\}_{i=1}^m$ of $E_{\mathcal{H}}$ and define $\Theta \in H_{B(\mathcal{E}, \mathbb{C}^m)}^{\infty}(\mathbb{D})$ by

$$\Theta(w) \eta = (\langle \eta, v_{1,w} \rangle_{\mathcal{E}}, \dots, \langle \eta, v_{m,w} \rangle_{\mathcal{E}}),$$

for all $\eta \in \mathcal{E}$ and $w \in \mathbb{D}$. Now

$$\|\Theta(w)^*x\| = \left\| \sum_{i=1}^m x_i v_{i,w} \right\| = \frac{1}{\|k_{\mathcal{M}}(\cdot, w)\|} \left\| \sum_{i=1}^m x_i \gamma_{i,w} \right\|,$$

and hence

$$\|\Theta(w)^*x\| \geq \delta \|x\|,$$

for all $x \in \mathbb{C}^m$ and $w \in \mathbb{D}$. Consequently, Θ is right invertible (cf. Proposition 3.7 in [20]). In particular,

$$\text{ran } M_{\Theta} = \mathcal{M} \otimes \mathbb{C}^m,$$

and since

$$M_{\Theta} \mathcal{S} = \{0\},$$

the module map $X : \mathcal{Q} \rightarrow \mathcal{M} \otimes \mathbb{C}^m$ defined by $Xf = \Theta f$ for all $f \in \mathcal{Q}$ is the required similarity. \blacksquare

Applying the preceding results we can generalize the similarity results in [13] and [8] where \mathcal{M} is assumed to be the Hardy module and the weighted Bergman modules of integer weights, respectively. However, the techniques involved in the proof are as same as those in [17], [13] and [8] and therefore, we only sketch a proof of it.

THEOREM 5.3. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be an \mathcal{M} -contractive Hilbert module. Then the following statements are equivalent:*

(i) \mathcal{H} is similar to $\mathcal{M} \otimes \mathbb{C}^m$.

(ii) There exists an anti-holomorphic pointwise invertible bundle map $\Phi : E_{\mathcal{M} \otimes \mathbb{C}^m} \rightarrow E_{\mathcal{H}}$ and $\delta > 0$ such that

$$\frac{1}{\delta} \|\eta_w\|_{E_{\mathcal{M} \otimes \mathbb{C}^m}(w)} \leq \|\Phi(w)\eta_w\|_{\mathcal{H}} \leq \delta \|\eta_w\|_{E_{\mathcal{M} \otimes \mathbb{C}^m}(w)},$$

for all $\eta_w \in E_{\mathcal{M} \otimes \mathbb{C}^m}(w)$ and $w \in \mathbb{D}$.

(iii) There exists a bounded solution φ defined on \mathbb{D} to the Poisson equation

$$\Delta \varphi(w) = \|\bar{\partial} \Pi(w)\|_2^2 - m |\mathcal{K}_{\mathcal{M}}(w)|.$$

(iv) The measure

$$(\|\bar{\partial} \Pi(w)\|_2^2 - m |\mathcal{K}_{\mathcal{M}}(w)|)(1 - |w|) dx dy$$

is Carleson and the estimate

$$(\|\bar{\partial} \Pi(w)\|_2^2 - m |\mathcal{K}_{\mathcal{M}}(w)|)^{\frac{1}{2}} \leq \frac{C}{1 - |w|}$$

holds for some $C > 0$.

Proof. The equivalence of (i) and (ii) is Theorem 5.2.

(ii) implies (iv): We note that

$$E_{\mathcal{M} \otimes \mathbb{C}^m}(w) = \ker(M_z - w)^* \cong k_{\mathcal{M}}(\cdot, w) \otimes \mathbb{C}^m,$$

and

$$E_{\mathcal{H}}(w) = \ker(M_z - w)^* \cong k_{\mathcal{M}}(\cdot, w) \otimes V(w).$$

Consequently, for a given bundle equivalence Φ from $E_{\mathcal{M} \otimes \mathbb{C}^m}$ to $E_{\mathcal{H}}$ there exists a bounded anti-holomorphic map $\Gamma : \mathbb{D} \rightarrow \mathcal{B}(\mathbb{C}^m, \mathcal{E})$ such that $\Phi(w)(k_{\mathcal{M}}(\cdot, w) \otimes \eta) = k_{\mathcal{M}}(\cdot, w) \otimes \Gamma(w)\eta$ and $V(w) = \text{ran} \Gamma(w)$ for all $\eta \in \mathbb{C}^m$ and $w \in \mathbb{D}$. Then the required conclusion readily follows from Proposition 0.5 in [17].

(iv) implies (iii): Assume that (iv) holds. Then the Green potential

$$\mathcal{G}(\lambda) := \frac{2}{\pi} \int_{\mathbb{D}} \int_{\mathbb{D}} \ln \left| \frac{w - \lambda}{1 - \bar{\lambda}w} \right| \|\bar{\partial} \Pi_V(w)\|^2 dx dy,$$

is uniformly bounded and hence there exists a bounded subharmonic function φ on \mathbb{D} such that $\Delta \varphi(w) \geq \|\bar{\partial} \Pi_V(w)\|^2$. The equality follows from the same argument as that of [8].

(iii) implies (i): We use Theorem 0.2 in [17] to get a bounded anti-holomorphic projection $\Theta(w)$ onto $\text{ran} \Pi_V(w)$. Let Θ_i be the inner part of the inner-outer factorization of $\Theta(w)$. Then it follows that Θ_i is invertible and the required similarity operator (see [13] or [8] for more details). \blacksquare

It is of interest to note the following consequence of Theorem 5.3: Let \mathcal{M} be a Cowen-Douglas atom and $\mathcal{H} \in B_m(\mathbb{D})$ be an \mathcal{M} -contractive Hilbert module and V be the hermitian anti-holomorphic vector bundle such that $E_{\mathcal{H}} \cong E_{\mathcal{M}} \otimes V$ (see Theorem 3.4).

Let $\tilde{\mathcal{M}}$ be another Cowen-Douglas atom and $\tilde{\mathcal{H}} \in B_m(\mathbb{D})$ be the $\tilde{\mathcal{M}}$ -contractive Hilbert module corresponding to the hermitian anti-holomorphic vector bundle $E_{\tilde{\mathcal{M}}} \otimes V$.

Now if \mathcal{H} is similar to $\mathcal{M} \otimes \mathbb{C}^m$ then by Corollary 4.2 and part (iii) of the previous theorem, we have

$$\Delta \varphi = \|\bar{\partial} \Pi_V(w)\|_2^2,$$

for some bounded subharmonic function on \mathbb{D} . Another application of Corollary 4.2 and part (iii) of the previous theorem to $\tilde{\mathcal{H}}$ yields the similarity of $\tilde{\mathcal{H}}$ to $\tilde{\mathcal{M}} \otimes \mathbb{C}^m$. Therefore, we have the following result.

COROLLARY 5.4. *Let \mathcal{M} and $\tilde{\mathcal{M}}$ be two Cowen-Douglas atoms. Then an \mathcal{M} -contractive Hilbert module \mathcal{H} corresponding to the hermitian anti-holomorphic vector bundle $E_{\mathcal{M}} \otimes V$ is similar to $\mathcal{M} \otimes \mathbb{C}^m$ if and only if the $\tilde{\mathcal{M}}$ -contractive Hilbert module $\tilde{\mathcal{H}}$ is similar to $\tilde{\mathcal{M}} \otimes \mathbb{C}^m$, where $\tilde{\mathcal{H}} \in B_m(\mathbb{D})$ is the Hilbert module corresponding to the hermitian anti-holomorphic frame $E_{\tilde{\mathcal{M}}} \otimes V$.*

The above result is a generalization of Corollary 4.5 (restricted to the Cowen-Douglas atoms) in [9] where the quotient module representations are assumed to be the orthocomplements of the submodules implemented by left invertible multipliers. Moreover, the free modules corresponding to the quotient modules are also assumed to be of finite rank.

6. CONCLUDING REMARKS

A number of questions and directions remain to be explored, including the similarity problems for the Dirichlet module. We point out that the notion of the Cowen-Douglas atom does not cover the Dirichlet module (see [16]). Some of the results of this paper can be generalized in the several variables set up. However, one of the key ideas to achieve results of full length is partly related to the corona problem in several variables (see [17]).

Another interesting question relates the quasi-affinity of Cowen-Douglas Hilbert modules. For the Hardy space, quasi-affinity to a submodule of a Hardy module is as same as the Hardy module it self. It is not known under what additional condition on the frame, that module will be quasi-affine to the Cowen-Douglas atom of finite multiplicity.

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